MATH 2050C Lecture 15 (Mar 9)

[Reminder: Midterm this Thursday - Friday.]

Last time ...... "Cauchy sequence"

Def^{n}: (x_n) Cauchy iff $\forall \varepsilon > 0, \exists H = H(\varepsilon) \in \mathbb{N}$ s.t. $|x_n - x_m| < \varepsilon \quad \forall n, m \geq H$

Remark: One do NOT need to refer to a limit value x in this def^{n}.

Thm (Cauchy Criteria) (x_n) Cauchy $\Leftrightarrow$ (x_n) convergent

Example: Let (x_n) be the sequence defined by

$$x_1 := 1 \quad ; \quad x_2 := 2 \quad ; \quad x_n := \frac{1}{2}(x_{n-1} + x_{n-2}) \quad \forall n \geq 3.$$

Show that (x_n) is convergent and find $\lim (x_n)$.

Think: $(x_n) := (1, 2, 1.5, 1.75, 1.625, \ldots\ldots)$

bdd, NOT monotone, ...

Pf: By M.I. (Exercise), we have

- $1 \leq x_n \leq 2 \quad \forall n \in \mathbb{N}$
- $|x_{n+1} - x_n| = \dfrac{1}{2^{n-1}} \quad \forall n \in \mathbb{N}$

Claim: (x_n) is "Cauchy"

Pf of Claim: Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $H \in \mathbb{N}$ s.t. $H > \dfrac{4}{\varepsilon}$.

Then, $\forall m, n \geq H$, we want to show

$$|x_m - x_n| < \varepsilon \quad \forall m, n \geq H$$

W.L.O.G. assume $m > n \geq H$.

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$= \frac{1}{2^{m-2}} + \frac{1}{2^{m-3}} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-n-1}}\right)$$

$$< \frac{1}{2^{n-1}} \cdot 2 = \frac{1}{2^{n-2}} \leq 4 \cdot \frac{1}{2^H} \leq 4 \cdot \frac{1}{H} < \varepsilon$$

By Cauchy Criteria, $\lim (x_n) =: x$ exists.

Consider the subseq. $(x_{2k-1})_{k \in \mathbb{N}}$

Note: $\lim\limits_{k \to \infty} (x_{2k-1}) = x$

$$x_{2k-1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{2k-3}}$$

$$= 1 + \frac{\frac{1}{2}\left(1 - \frac{1}{4^{k-1}}\right)}{1 - \frac{1}{4}}$$

Take $k \to \infty$. we have $x = 1 + \dfrac{1/2}{3/4} = \dfrac{5}{3}$ #

$x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$

Take $n \to \infty$,

$x = \frac{1}{2}(x + x) = x$

not helpful

Course Outline

- real number system $\mathbb{R}$
- limit of seq. (x_n)

} preparations

- limit of function, ie "$\lim\limits_{x \to a} f(x) = L$"
- "continuity" of functions

Limit of Functions (Ch. 4 in textbook)

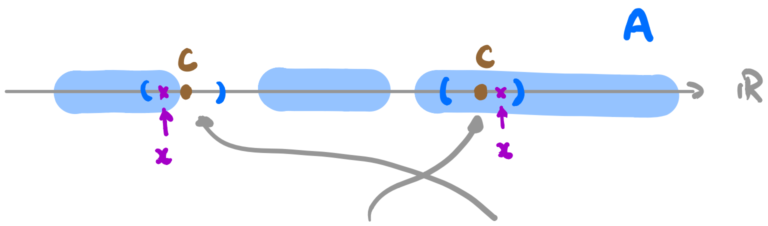
GOAL: Define $\lim_{x \to c} f(x) = L$ for functions $f : A \subseteq \mathbb{R} \to \mathbb{R}$

We shall only define $\lim_{x \to c} f(x)$ for those "c"'s which are "cluster point" of A.

so $f(x)$ is defined.

IDEA: $f(x) \overset{\varepsilon}{\approx} L$ when $x \overset{\delta}{\approx} c$ and $x \in A$

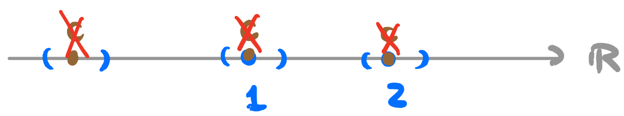
Def$^{\underline{n}}$: Let $A \subseteq \mathbb{R}$. We say that $c \in \mathbb{R}$ is a cluster point of A iff $\forall\, \delta > 0$, $\exists\, x \in A$ s.t. $x \neq c$ and $|x - c| < \delta$



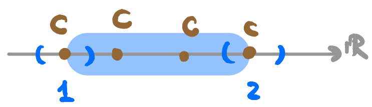
Remark: A cluster pt. $c \in \mathbb{R}$ may or may not belong to A.

Examples:

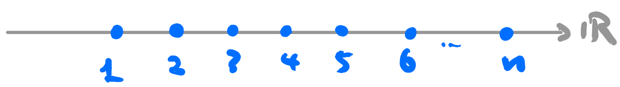
- $A = \{1, 2\}$ NO cluster pt.



- $A = (0, 1)$ Any $c \in [0, 1]$ is a cluster pt

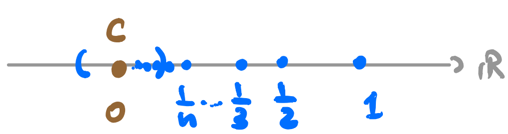


- $A = \{a_1, \dots, a_n\}$ NO cluster pt.
- $A = \mathbb{N}$ NO cluster pt.



- $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ ONLY 1 cluster pt

  $c = 0$



Prop: $c \in \mathbb{R}$ is a cluster point of A

$\iff \exists$ seq. (a_n) in A s.t. $a_n \neq c$ $\forall n \in \mathbb{N}$

and $\lim (a_n) = c$

Sketch of Proof: $(\Rightarrow)$ Take $\delta_n := \frac{1}{n}$, by def^n. $\exists$ $a_n \in A$ st

$a_n \neq c$ and $|a_n - c| < \delta_n = \frac{1}{n} \xrightarrow{\text{as } n\to\infty} 0$

Common mistake in Ex. 3.3.7

$x_1 := a > 0$

$x_{n+1} := x_n + \frac{1}{x_n}$ $\forall n \in \mathbb{N}$.

Assume $\lim (x_n) =: x$ exist. $\underset{\text{Correct unless } x \neq 0.}{\overset{\text{not}}{\Rightarrow}}$ $x = x + \frac{1}{x}$ $\Rightarrow$ $0 = \frac{1}{x}$ $\times$.

We now state the most important definition for this chapter.

Def^n: Let $f : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function.

Suppose $c \in \mathbb{R}$ is a cluster point of A.

We say that "f converges to $L \in \mathbb{R}$ at c", written

"$\lim_{x \to c} f(x) = L$" or "$f(x) \to L$ as $x \to c$"

iff $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$|f(x) - L| < \varepsilon$, $\forall x \in A$ where $\underbrace{0 < |x - c|}_{\text{so } x \neq c} < \delta$

Example 1: Let $f: \mathbb{R} \to \mathbb{R}$ be $f(x) := x$ for all $x \in \mathbb{R}$.

$$\lim_{x \to c} f(x) = c \qquad \forall\, c \in \mathbb{R}.$$

Pf: Any $c \in \mathbb{R}$ is a cluster pt. of $A = \mathbb{R}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $\delta > 0$ st $\delta = \varepsilon$

THEN, $\forall\, x \in \mathbb{R}$, and $0 < |x - c| < \delta$, we have

$$|f(x) - c| = |x - c| < \delta = \varepsilon$$

Remark: $\lim_{x \to c} f(x)$ may exists with f being defined at c.

E.g) $f: A = (0,1) \to \mathbb{R}$; $f(x) := x$

$$\Rightarrow \quad \lim_{x \to 1} f(x) = 1 \notin A$$

Example 2: $\boxed{\lim_{x \to c} x^2 = c^2}$ i.e. $f: A = \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

Pf: Fix $c \in \mathbb{R}$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Note: Suppose $|x - c| < 1$, then

$$|x| \le |x - c| + |c| < 1 + |c|$$

Choose $\delta := \min\left\{1, \dfrac{\varepsilon}{2(1 + 2|c|)}\right\}$

if $0 < |x - c| < \delta$, then

$$|x^2 - c^2| = |x + c| \cdot |x - c|$$
$$\le (|x| + |c|) \cdot |x - c|$$
$$\le (2|c| + \delta) \cdot \delta < \varepsilon.$$

$|x - c| < \delta \Rightarrow |x| < |c| + \delta$

THEN, $\forall x \in A = \mathbb{R}$ with $0 < |x - c| < \delta$, we have

$$|x^2 - c^2| = |x+c| \cdot |x - c| \leq (1 + 2|c|)\,\delta < \varepsilon$$

$\blacksquare$

Example 3: $\boxed{\lim_{x \to c} \frac{1}{x} = \frac{1}{c}}$ where $c \neq 0$.

Considering $f : A = \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) := \frac{1}{x}$

Note: Any $c \in \mathbb{R}$ is a cluster pt of A

Pf: Let's assume $c > 0$.

Let $\varepsilon > 0$ be fixed but arbitrary.

Note: If $|x - c| < \frac{c}{2}$, then

$$|x| > \frac{c}{2} > 0$$

Take $\delta := \min\left\{ \frac{c}{2}, \frac{\varepsilon c^2}{2} \right\} > 0$.

Then, $\forall x \in A$ and $0 < |x - c| < \delta$

we have

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{1}{|x|} \cdot \frac{1}{|c|} \cdot |x - c| < \frac{2}{c^2} \cdot \delta \leq \varepsilon$$

$\blacksquare$

If $0 < |x - c| < \delta$, then

$$\left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x - c|}{|xc|}$$

$$= \frac{1}{|x|} \frac{1}{|c|} |x - c| < \frac{2}{c^2} \cdot \delta \leq \varepsilon$$

Need $|x|$ bdd away from 0.